Single-Lifting Macaulay-Type Formulae of Generalized Unmixed Sparse Resultants

Ioannis Z. Emiris
Dept Informatics & Telecoms
University of Athens, Greece
emiris@di.uoa.gr

Christos Konaxis
Dept Informatics & Telecoms
University of Athens, Greece
ckonaxis@di.uoa.gr

ABSTRACT

Resultants are defined in the sparse (or toric) context in order to exploit the structure of the polynomials as expressed by their Newton polytopes. Since determinantal formulae are not always possible, the most efficient general method for computing resultants is as the ratio of two determinants. This is made possible by Macaulay's seminal result [15] in the dense homogeneous case, extended by D'Andrea [6] to the sparse case. However, the latter requires a lifting of the Newton polytopes, defined recursively on the dimension. Our main contribution is a single lifting function of the Newton polytopes, which avoids recursion, and yields a simpler algorithm for computing Macaulay-type formulae of sparse resultants, in the case of generalized unmixed systems, where all Newton polytopes are scaled copies of each other. In the mixed subdivision used to construct the matrices, our algorithm defines significantly fewer cells than D'Andrea's, and is easier to implement and analyze, though the matrices are same in both cases. Our approach provably extends to mixed systems of up to 4 polynomials, and those whose Newton polytopes have a sufficiently different face structure, but it should be generalizable to any mixed system. Our Maple implementation is applied to study a full example.

Keywords

Sparse resultant, Macaulay formula, Minkowski sum, mixed subdivision, generalized unmixed system

1. INTRODUCTION

Resultants are fundamental constructions for studying and solving algebraic systems; for instance, they reduce system solving to linear algebra or to factoring univariate polynomials. The sparse (or toric) resultant captures the structure of the polynomials by combinatorial means and constitutes the cornerstone of sparse elimination theory [3, 13].

The resultant is defined for a system of $n+1$ polynomials in n variables over coefficient ring K. It is the unique, up to sign, integer polynomial over K which vanishes precisely when the system has a common root. The *classical, or projective,* resultant expresses solvability of a system of dense polynomials $f_i \in K[x_1, \ldots, x_n]$ in $\mathbb{P}^n$ over the algebraic closure $\overline{K}$. The *sparse, or toric,* resultant expresses solvability of a system of Laurent polynomials $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over a toric variety X, defined by the supports of f_i, s.t. $(\overline{K}^*)^n$ is a dense subset of X.

A resultant is most efficiently expressed by a *resultant matrix:* this is generically nonsingular, its determinant is a multiple of the resultant, and the determinant's degree wrt the coefficients of one polynomial equals that of the resultant. For $n = 1$ there are resultant matrices, named after Sylvester and Bézout, whose determinant equals the resultant. Unfortunately, such determinantal formulae do not generally exist for $n > 1$, except for specific cases, e.g. [7, 9, 10, 14]. Macaulay's seminal result [15] expresses the extraneous factor as a minor of the resultant matrix, for classical resultants of dense homogeneous systems, thus yielding the most efficient general method for computing such resultants.

Resultant matrices for the sparse resultant were first constructed in [1]. The construction relies on a lifting of the given polynomial supports, which defines a mixed subdivision of their Minkowski sum into mixed and non-mixed cells, then applies a perturbation δ so as to define the integer points that index the matrix. The algorithm was extended in [2, 4, 17]. In the case of dense systems, the matrix coincides with Macaulay's numerator matrix.

Extending the Macaulay formula to sparse resultants had been conjectured in [2, 3, 11, 13, 17]; it was a major open problem in elimination theory. We cite [17, p.219], where $P_{\omega,\delta}$ is the extraneous factor, and ω denotes the lifting: *"It is an important open problem to find a more explicit formula for $P_{\omega,\delta}$ in the general sparse case. [...] This problem is closely related to the following empirical observation. For suitable choice of δ and ω, the matrix $\mathcal{M}_{\delta,\omega}$ seems to have a block structure which allows to extract the resultant from a proper submatrix. "*

D'Andrea's fundamental result [6] answers the conjecture by a *recursive* definition of a Macaulay-type formula, cf sec. 3. But this approach does not offer a global lifting, in order to address the stronger original conj. 1. Let M be the resultant matrix, also known as Newton matrix, and $M^{(nm)}$ its submatrix indexed by points in non-mixed cells of the mixed subdivision.

CONJECTURE 1. *[11, Conj.3.1.19] [2, Conj.13.1] There exist perturbation vector δ and $n+1$ lifting functions for which the determinant of matrix $M^{(nm)}$ divides exactly the*

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.
Copyright 200X ACM X-XXXXX-XX-X/XX/XX ...$10.00.

determinant of Newton matrix M and, hence, the sparse resultant of the given polynomial system is $\det M / \det M^{(nm)}$.

Our main contribution is to give an affirmative answer to this stronger conjecture by presenting a single lifting which constructs Macaulay-type formulae for generalized unmixed systems, i.e. when all Newton polytopes are scaled copies of each other. We state our main result, to be proven in sec. 4:

THEOREM 2. *The global lifting of sec. 2 produces a Macaulay-type formula for the sparse resultant of a system of polynomials with scaled Newton polytopes.*

Our algorithm is generalized, in sec. 5, to certain mixed systems: those with $n \leq 3$, and reduced systems, defined in [18] to possess sufficiently different Newton polytopes. Most of these cases have been studied: reduced systems were settled in [5], and bivariate systems ($n = 2$) in [8], by directly establishing the extraneous factor. Our approach should eventually make the single-lifting algorithm applicable to the fully general case.

Using a unique lifting function essentially means that we consider a deformed system, defined by adding a new variable t so that each input monomial x^a gets multiplied by t^b, where $b \in \mathbb{Z}$ is the lifting value of $a \in \mathbb{Z}^n$. Such deformations capture the system's behavior at toric infinity, hence lie at the heart of most theorems in sparse elimination (e.g. sparse homotopies, sparse resultants, the sparse Nullstellensatz). Such combinatorial methods consitute one of the two main approaches for studying sparse resultants, e.g. [2, 3, 7, 16, 17], the other relying on Koszul complexes and their generalizations, e.g. [9, 10, 14].

D'Andrea's [6] recursive construction requires one to associate integer points with cells of every dimension from n to 1. Our algorithm constructs the resultant matrix directly, without recursion, by examining only n-dimensional cells. These are more numerous than the n-dimensional cells in [6] but our algorithm defines significantly fewer cells totally, and is overall simpler, which is important for implementing and analyzing the algorithm. The disadvantage of our method is to consider extra points besides the input supports.

Existing public-domain Maple implementations cover only the original Canny-Emiris construction [2], either standalone[1] or as part of library Multires[2]. We have implemented this paper's algorithm in Maple; it is available upon request by the authors.

The rest of the paper is structured as follows. The next section introduces some necessary notions, and defines the single lifting that produces Macaulay-type formulae. Sec. 3 recalls the recursive algorithm of [6], and sec. 4 proves the equivalence of the two constructions. Sec. 5 sketches the extension of our algorithm to mixed systems. Sec. 6 analyzes the complexity of both algorithms. The appendix offers a full example computed by our implementation.

2. SINGLE LIFTING CONSTRUCTION

For any polytopes or point sets A, B, let $\langle A \rangle$ denote the affine span (or hull) of A over $\mathbb{R}$ and $\langle A, B \rangle$ the affine span of $A \cup B$ over $\mathbb{R}$.

Let the polynomials' supports be $A_0, \dots, A_n \subset \mathbb{Z}^n$ with Newton polytopes

$$Q_0, \dots, Q_n \subset \mathbb{R}^n, \; Q_i = \mathrm{CH}(A_i),$$

where $\mathrm{CH}(\cdot)$ denotes convex hull. As matrix construction algorithms typically do, we define a regular and fine (or tight) mixed subdivision of the Minkowski sum $\sum_{i=0}^n Q_i$; cf [3, 13]. Regularity implies the subdivision is in bijective correspondence with the face structure of the upper (or lower) hull of the Minkowski sum of $Q_0, \dots, Q_n$ after they are lifted to $\mathbb{R}^{n+1}$. Each cell in $\mathbb{R}^n$ is written uniquely as the Minkowski sum of faces F_i of the Q_i. A fine subdivision is characterized by an equality between cell dimension and the sum of the faces' dimensions. We focus on cells of maximal dimension n, and call them maximal or, simply, cells. We distinguish them as mixed and non-mixed: the former are the Minkowski sum of n edges and a vertex. Mixed cells are i-mixed if this vertex lies in A_i. The *type* of a cell is either i-mixed or non-mixed.

The Minkowski sum $\sum_{i=0}^n Q_i$ is perturbed by a sufficiently small and in sufficiently generic position vector $\delta \in \mathbb{Q}^n$. Let Z be the integer lattice generated by $\sum_{i=0}^n A_i$. The lattice points in $\mathcal{E} = Z \cap (\sum_{i=0}^n Q_i + \delta)$ are associated to a unique maximal cell of the subdivision, and this allows us to construct a resultant matrix M whose rows and columns are indexed by these points.

Definition 1. Let $p \in \mathcal{E}$ lie in a cell $F_0 + \cdots + F_n + \delta$ of the perturbed mixed subdivision, where F_i is a face of Q_i. The *row content (RC)* of p is (i, j), if $i \in \{0, \dots, n\}$ is the largest integer such that F_i equals a vertex $a_{ij} \in A_i$.

The main idea of both our and D'Andrea's algorithms is that one point, say $b_{01} \in Q_0$, is lifted significantly higher. Then, the 0-th summand of all maximal cells is either b_{01} or a face not containing it. In D'Andrea's case, facets not containing b_{01} correspond to different subsystems where the algorithm recurses (each time on the integer lattice specified by that subsystem). In designing a unique lifting, the issue is that points appearing in two of these subsystems may be lifted differently in different recursions. To overcome this, we introduce several points c_{il}, for different l, very close (wrt $\mathbb{Z}$) to every b_{ij}, which is lifted very high at recursion i by D'Andrea's algorithm. This captures the different roles b_{ij} may assume.

Algorithm B. Our algorithm uses $\mathcal{E}$ to index the rows (and columns) of the numerator matrix of our Macaulay-type formula. In particular, polynomial $x^{p-a_{ij}} f_i$ fills in the row indexed by the lattice point p in def. 1. We now focus on generalized unmixed systems, where

$$Q_i = k_i Q \subset \mathbb{R}^n,$$

for some n-dimensional lattice polytope Q and $k_i \in \mathbb{N}^*, i = 0, \dots, n$. Then, the denominator shall be indexed by points lying in non-mixed cells.

Definition 2. For $i = 0, \dots, n-1$, and any $(n-i)$-dimensional face $k_i F_{ij} \subset Q_i$, where j ranges over all such faces, let $\delta_{ij} \in \mathbb{Q}^n$ denote a *perturbation* vector s.t.:

(1) it lies in the relative interior of $k_i F_{ij}$,

(2) it is sufficiently small compared to lattice Z, and $\|\delta_{ij}\| \ll \|\delta\|$, where $\|\cdot\|$ is Euclidean norm,

(3) it is sufficiently generic to avoid all edges in the mixed subdivision of $\sum_{i=0}^n Q_i$.

Let b_{ij}, for some valid $j > 1$, be vertex of Q_i. We shall use the perturbation vectors of def. 2 to define additional points *not* contained in the input supports.

[1] http://www.di.uoa.gr/∼emiris/soft_alg.html

[2] http://www-sop.inria.fr/galaad/logiciels/multires.html

Definition 3. Alg. B defines points $c_{ij} \in Q_i \cap \mathbb{Q}^n$. First, $c_{01} := b_{01} + \delta_{01}$. For $i = 0, \ldots, n-2$ and any F_{ij} as in def. 2, choose facets $F_{(i+1)h} \subset F_{ij}$ s.t.:

(1) $k_i F_{(i+1)h}$ does *not* contain b_{ij}, and

(2) $k_{i+1} F_{(i+1)h}$ does *not* contain any of the already defined $c_{(i+1)l}$'s.

For each such facet set: $c_{(i+1)h} := b_{(i+1)h} + \delta_{(i+1)h}$.

The previous definition implies a many-to-one mapping from the set of c_{ij}'s to that of b_{ij}'s: It reduces to a bijection when restricted to a fixed face $F_{ij} \subset Q_i$ containing b_{ij}. Condition 1 of def. 2 implies that c_{ij} does not lie on a face of dimension $< n-i$ and lies in the interior of $(n-i)$-dimensional F_i. We can reduce the number of the c_{ij}'s in alg. B, but this would complicate the subsequent proofs.

For an application of def. 3 when $n = 2$, see fig. 1a, where we define points c_{ij} also on edges.

Definition 4. Let $h_0 \gg h_1 \gg \ldots \gg h_{n-1} \gg 1$. Alg. B uses sufficiently random linear functions $H_i, i = 0, \ldots, n$, s.t.

$$1 \gg H_i(a_{ij}) > 0, \text{ and } H_i \gg H_t, \ i < t,$$

where $a_{ij} \in A_i$ and $i, t = 0, \ldots, n, \ j = 1, \ldots, |A_i|$. Alg. B defines global lifting β as follows:
(1) $c_{ij} \mapsto h_i^{n-i}, \ c_{ij} \in k_i F_{ij} \subset Q_i, \ i = 0, \ldots, n-1$; this is called primary lifting.
(2) $a_{ij} \mapsto H_i(a_{ij}), \ a_{ij} \in A_i, i = 0, \ldots, n$.

Let F^β denote face F lifted under β. Now c_{tj}^β, for all valid j, is much higher, resp. lower, than any $c_{ij}^\beta, i > t$, resp. $i < t$. The β-induced subdivision contains edges with one or two vertices among the c_{ij}, and edges from the Q_i. The vertex set of the upper hull of Q_i^β contains some or all of the c_{ij}^β and the lifted vertices of Q_i.

When all Q_i are simplices, as in the classical dense case, it suffices to apply a primary lifting to one point per Q_i. Thus our scheme generalizes the approach by Macaulay [15].

The resultant matrix constructed by alg. B is indexed by all lattice points in $\mathcal{E}$. To decide the content of each row, every point is associated to a unique (maximal) cell of the mixed subdivision according to def. 1. The t-mixed cells contain lattice points as follows:

$$p \in k_0 E_0 + \cdots + k_{t-1} E_{t-1} + c_{tj} + k_{t+1} E_{t+1} + \cdots + k_n E_n \cap Z$$

for edges $E_i \subset Q$ spanning $\mathbb{R}^n$. This gives optimal writing

$$p = p_0 + \cdots + p_{t-1} + (b_{tj} + \delta_{tj}) + p_{t+1} + \cdots + p_n, \ p_i \in A_i \cap E_i.$$

Hence, the row indexed by p, as with matrix constructions in [2, 6], contains a multiple of $f_t(x)$:

$$x^{p_0 + \cdots + p_{t-1} + p_{t+1} + \cdots + p_n} f_t(x),$$

and the diagonal element is the coefficient of b_{tj} in $f_t(x)$. Similarly, for the rows corresponding to lattice points in non-mixed cells.

3. RECURSIVE CONSTRUCTION

We discuss D'Andrea's recursive construction of a Macaulay-type formula [6]. There are certain free parameters in the algorithm which we specify so as to obtain a version very similar to our approach.

At the input of the 0-step the algorithm may use an additional polytope mQ, for any $m \in \mathbb{R}$, which we omit.We describe the t-th recursive step, for $t = 0, 1, \ldots, n-1$.

Algorithm A. The input are polytopes

$$l_0 P^{(t)}, \ldots, l_{t-1} P^{(t)}, k_t P^{(t)}, \ldots, k_n P^{(t)} \subset \mathbb{R}^{n-t}, \ l_i \in [0, k_i] \cap \mathbb{Q},$$

the integer lattice $L^{(t)}$ spanned by $\sum_{i=t}^n A_i \cap k_i P^{(t)}$, and perturbation vector $\delta_t \in \mathbb{Q}^{n-t}$. Here, $k_i P^{(t)}, \ i \geq t$, is an $(n-t)$-dimensional face of $k_i Q$, thus $P^{(0)} = Q$. Also, $P^{(t)}$ is a facet of $P^{(t-1)}$, and $l_i P^{(t)}, \ i < t$, is homothetic to $k_i P^{(t)}$. These constructions shall be defined at the Recursion Phase. Also, $L^{(0)} = Z$ and $\delta_0 = \delta$.

Construction Phase: Vertex $b_{tj} \in k_t P^{(t)} \cap A_t$, is lifted to 1. We require that $b_{tj} = c_{tj} - \delta_{tj}$. All other vertices of all input polytopes are lifted to 0. This is the primary lifting which partitions the Minkowski sum of the input polytopes into a primary cell

$$l_0 P^{(t)} + \cdots + l_{t-1} P^{(t)} + b_{tj} + k_{t+1} P^{(t)} + \cdots + k_n P^{(t)} + \delta_t, \quad (1)$$

of dimension $n-t$, and several secondary cells. Each secondary cell is defined by an inner normal $v \in \mathbb{Q}^{n-t}$ to a facet of $k_t P^{(t)}$ not containing b_{tj}.

Polytopes $\sum_{i=0}^{t-1} l_i P^{(t)}, k_{t+1} P^{(t)}, \ldots, k_n P^{(t)}$ are lifted by applying the restriction of β on them. We consider β fixed throughout the algorithm. The upper hull of the Minkowski sum of the lifted polytopes induces a mixed subdivision of $\sum_{i=0}^{t-1} P^{(t)} + k_{t+1} P^{(t)} + \cdots + k_n P^{(t)}$, which is then perturbed by δ_t. The lattice points p of $L^{(t)}$ contained in the perturbed subdivision, are assigned RC by def. 1. This also assigns RC to points $p + b_{tj}$ contained in the intersection of (1) with $L^{(t)}$. Let us take care of the c_{ij}. If point p lies in

$$(F + F_{t+1} + \cdots + F_n + \delta_t) \cap L^{(t)}, \quad (2)$$

where $F_i \subset k_i Q_i, \ i > t, \ F \subset \sum_{i=0}^{t-1} l_i P^{(t)}$, having RC$(p) = (h, j)$, where $F_h = c_{hj} = b_{hj} + \delta_{hj}$, then the corresponding matrix row is filled in by $x^{p - b_{hj}} f_h$.

Face $F \subset \sum_{i=0}^{t-1} P^{(t)}$ in (2), can be analyzed as $F = l_0 F_0 + \cdots + l_{t-1} F_{t-1}$, where $F_i \subset P^{(t)}$ for $i < t$. Moreover, every cell in (1) is the Minkowski sum of b_{tj} and the cell in (2).

Mixed cells of type 0 are defined here as in sec. 2. A t-mixed cell wrt alg. A, for $t > 0$, shall have $n-t$ linear summands from polytopes $k_{t+1} P^{(t)}, \ldots, k_n P^{(t)}$ and a 0-dimensional summand from polytope $\sum_{i=0}^{t-1} l_i P^{(t)}$. This summand can be analyzed as $l_0 p_0 + \cdots + l_{t-1} p_{t-1}$, where $p_i \in P^{(t)}$, for $i = 0, \ldots, t-1$ and $l_i p_i$ stands for a scalar multiple of p_i, seen as a vector. This leads to:

LEMMA 3. *The maximal cells at step t of alg. A are, for some j and $l_i \in [0, k_i]$, of the form:*

$$l_0 F_0 + \cdots + l_{t-1} F_{t-1} + b_{tj} + k_{t+1} F_{t+1} + \cdots + k_n F_n + \delta_t, \quad (3)$$

where F_i is the projection of a face of the upper hull of $P^{(t)}$ lifted by β, $\dim(\langle F_0, \ldots, F_{t-1}, F_{t+1}, F_n \rangle) = n-t$. Specifically, the t-mixed cells in alg. A are:

$$l_0 p_0 + \cdots + l_{t-1} p_{t-1} + b_{tj} + k_{t+1} E_{t+1} + \cdots + k_n E_n + \delta_t, \quad (4)$$

where $E_{t+1}, \ldots, E_n$, are projections of edges on the upper hull of $P^{(t)}$ lifted by β, $\dim(\langle E_{t+1}, \ldots, E_n \rangle) = n-t$, and points $p_i \in P^{(t)}$, for $i = 0, \ldots, t-1$.

Recursion Phase: When $t = n-1$, the algorithm terminates, since it has reached the Sylvester case. Otherwise, it recurses: let $P^{(t+1)}$ be the facet of $P^{(t)}$ supported by v.

The (perturbed) secondary cell corresponding to v is

$$\mathcal{F}_v = l_0P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + \mathrm{CH}(b_{tj}, k_tP^{(t+1)}) + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_t. \quad (5)$$

Its associated diameter is

$$d_v = b_{tj} \cdot v - \min_{p \in \mathrm{CH}(b_{tj}, k_tF)} \{p \cdot v\} \in \mathbb{N}^*,$$

where $\cdot$ stands for inner product. We define two sublattices of $L^{(t)}$: $L_+^{(t)}$ is spanned by $\sum_{i=t+1}^{n} A_i \cap k_iP^{(t+1)}$ and $L_v^{(t)}$ is the sublattice orthogonal to v. They have the same dimension, so we define the (finite) index $\mathrm{ind}_v = [L_v^{(t)} : L_+^{(t)}]$, equal to the quotient of the volumes of their base cells. Let q range over the ind_v coset representatives for $L_+^{(t)}$ in $L_v^{(t)}$.

Let $l_t \in [0, k_t]$ take d_v distinct values corresponding to different values of $p \cdot v$ for all $p \in (\mathrm{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap L^{(t)}$. Note that $l_tP^{(t+1)}$ is homothetic to $k_tP^{(t+1)}$. Let $\delta_t' \in \mathbb{Q}^{n-t}$ be a translation vector such that $l_tP^{(t+1)} + \delta_t'$ contains at least one point in $(\mathrm{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap L^{(t)}$.

In particular, $l_tP^{(t+1)} + \delta_t'$ equals $k_tP^{(t+1)}$ iff $l_t = k_t$, and vertex b_{tj} iff $l_t = 0$, otherwise it equals $(\mathrm{CH}(b_{tj}, k_tP^{(t+1)}) + \delta_t) \cap H$, where H is a hyperplane parallel to a supporting hyperplane of $k_tP^{(t+1)}$;cf [6, lem.3.3]. By abuse of notation, in the rest of this paper we shall denote H, and the supporting hyperplanes of faces $k_tP^{(t+1)}$ and b_{tj} of the previous convex hull, as $\langle l_tP^{(t+1)} \rangle$.

Points in $(\mathcal{F}_v + \delta_t) \cap L^{(t)}$ are partitioned into d_v subsets (one per value of l_t), called *slices*, of the form

$$l_0P^{(t+1)} + \cdots + l_{t-1}P^{(t+1)} + (l_tP^{(t+1)} + \delta_t') + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_t \cap L^{(t)}, \quad (6)$$

which can be rearranged as

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_\lambda \cap L^{(t)}, \quad (7)$$

where $\delta_\lambda = \delta_t + \delta_t'$. Moreover, δ_λ can be decomposed as $\delta_\lambda^v + \delta_{\lambda v}$, where $\delta_\lambda^v \in \mathbb{Q}v$ and $\delta_{\lambda v} \in L_+^{(t)} \otimes \mathbb{Q}$. Now, every point in (7) corresponds to a point in

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_{\lambda v} \cap (q + L_+^{(t)}),$$

for some coset representative q. Set $\delta_{t+1} := \delta_{\lambda v} - q$, $L^{(t+1)} := L_+^{(t)}$, and observe that point p belongs in (7) iff point

$$p' := p - \delta_\lambda^v - q \quad (8)$$

belongs in

$$l_0P^{(t+1)} + \cdots + l_tP^{(t+1)} + k_{t+1}P^{(t+1)} + \cdots + k_nP^{(t+1)} + \delta_{t+1} \cap L^{(t+1)}. \quad (9)$$

We call this set a *piece*; δ_{t+1} carries the information to define the piece from the input polytopes and $L^{(t+1)}$. The algorithm recurses on each of the ind_v such pieces. The set

$$l_0P^{(t+1)}, \ldots, l_tP^{(t+1)}, k_{t+1}P^{(t+1)}, \ldots, k_nP^{(t+1)}, \ \delta_{t+1}$$

over $L^{(t+1)}$ is exactly like the original input, only one dimension lower. This completes the algorithm.

Remark 1. Since every point p' in a piece corresponds bijectively to a point p in a slice via the monomial bijection (8), we shall often consider a piece as a subset of a slice and omit the translation.

At the end of the recursion, RC is defined on $\mathcal{E}$. Alg. A defines a partition of $\mathcal{E}$ in the form of a collection of mixed subdivisions of primary cells (of decreasing dimension). The 1-summands from Q_i in the cells are defined by any point in A_i or among the c_{ij}, for all valid j, and may be multiplied by a rational number in $(0, k_i]$.

4. EQUIVALENCE OF CONSTRUCTIONS

The single-lifting algorithm is alg. B; its overall effect is very similar to that of alg. A, since they both use β. The former partitions $\mathcal{E}$ into sets of points in n-dimensional cells and assigns RC, whereas, as we show in the next lemmas, alg. A partitions $\mathcal{E}$ into subsets which, at step t, lie on the intersection of a $(n-t)$-dimensional hyperplane with an n-dimensional cell of β. Note that the intersection itself, as a subset of $\mathbb{R}^{n-t}$ does not coincide with the cell of alg. A. However, their set difference is of infinitesimal volume. Although both algorithms use β to subdivide their input polytopes, they do so in a different fashion; alg. B applies β to every Q_i, whereas alg. A does so recursively to a different set of polytopes at every step.

In the rest of the paper, for simplicity, we shall omit the translation vectors δ_t. Moreover, unless otherwise stated, we shall treat every slice and piece as a polytope and not as the set of points in the intersection of this polytope with an appropriate lattice. In particular, we shall be interested only on the form a slice\piece as a Minkowski sum of polytopes. The existence of a translation vector, so as this polytope contains integer points in the lattice under consideration, shall be implied.

We now establish the correspondence between the two algorithms for $t = 0$, then generalize to arbitrary t. At step 0 of alg. A, b_{01} is lifted to 1 while every other vertex of all input polytopes to 0; this creates primary cell

$$pr.cell_0^{(A)} := b_{01} + k_1Q + \cdots + k_nQ,$$

and several secondary cells of the form

$$sec.cell_0^{(A)} := \mathrm{CH}(b_{01}, k_0P^{(1)}) + k_1P^{(1)} + \cdots + k_nP^{(1)},$$

each corresponding to a facet $P^{(1)}$ of Q not containing b_{01}. In alg. B, c_{01} plays the role of b_{01} and this leads to a group of cells covering the corresponding primary cell

$$pr.cell_0^{(B)} := c_{01} + k_1Q + \cdots + k_nQ,$$

and several groups of cells, each group covering

$$sec.cell_0^{(B)} := \mathrm{CH}(c_{01}, k_0P^{(1)}) + k_1P^{(1)} + \cdots + k_nP^{(1)},$$

which is a typical n-dimensional secondary cell wrt alg. B.

Remark 2. All cells within $pr.cell_0^{(A)}$ and $pr.cell_0^{(B)}$ differ only at their first summand; the former are of the form $b_{01} + F_1 + \cdots + F_n$, whereas the latter are $c_{01} + F_1 + \cdots + F_n$, where F_i is a face of Q_i, since β is used by both algorithms to subdivide $Q_1 + \cdots + Q_n$, and $c_{01} = b_{01} + \delta_{01}$.

LEMMA 4. *$pr.cell_0^{(A)} \cap \mathcal{E} = pr.cell_0^{(B)} \cap \mathcal{E}$, and points in this set are assigned the same RC under both algorithms.*

PROOF. Recall that $\delta_0 = \delta$ and consider the subdivision of $\sum_{i=0}^{n} Q_i$ induced by β and compare $pr.cell_0^{(A)} + \delta$ and $c_{01} + Q_1 + \cdots + Q_n + \delta = b_{01} + \delta_{01} + Q_1 + \cdots + Q_n + \delta$. These polytopes differ by δ_{01}, which is very small. Moreover, by the choice of δ, the boundary of $pr.cell_0^{(A)} + \delta$ has no points in Z. Since, by def. 2, $\|\delta\| \gg \|\delta_{01}\|$, the two polytopes contain the same Z-points. This settles the first claim.

The second claim follows from rem. 2 and the fact that the two subdivisions may only differ in cells having vertex b_{01} instead of c_{01}. Since $c_{01} - b_{01} = \delta_{01}$ is very small compared to Z, even these cells contain the same Z-points. □

Each $sec.cell_0^{(A)}$ is divided by alg. A into slices $l_0 P^{(1)} + k_1 P^{(1)} + \cdots + k_n P^{(1)}$, one for each value of $l_0 \in [0, k_0]$. Each slice is partitioned into pieces on which alg. A recurses producing $(n-1)$-dimensional primary cell

$$pr.cell_1^{(A)} := l_0 P^{(1)} + b_{1j} + k_2 P^{(1)} + \cdots + k_n P^{(1)}, \qquad (10)$$

and secondary cells

$$sec.cell_1^{(A)} := l_0 P^{(2)} + CH(b_{1j}, k_1 P^{(2)}) + k_2 P^{(2)} + \cdots + k_n P^{(2)}. \qquad (11)$$

Every piece of a given slice lies on lattice $L^{(1)}$ and can be thought of as the intersection of a translation of that slice, regarded as a polytope, with $L^{(1)}$. Recall that, by rem. 1, we shall consider a piece as subset of a slice.

Similarly to alg. A, we can partition the corresponding $sec.cell_0^{(B)}$ into slices:

$$l'_0 P^{(1)} + k_1 P^{(1)} + \cdots + k_n P^{(1)},$$

by intersecting $\mathrm{CH}(c_{01}, k_0 P^{(1)})$ with a hyperplane parallel to (a supporting hyperplane of) $k_0 P^{(1)}$. Recall that we denote this hyperplane as $\langle l'_0 P^{(1)} \rangle$.

Remark 3. Observe that each slice of $sec.cell_0^{(B)}$ (resp. $sec.cell_0^{(A)}$) parameterized by l'_0 (resp. l_0), is homothetic to a facet of this secondary cell, supported by $\langle k'_0 P^{(1)} \rangle$ (resp. $\langle k_0 P^{(1)} \rangle$). Moreover, this homothecy is defined by a homothecy only on the first summand $k_0 P^{(1)}$ of this facet.

Hyperplanes $\langle l'_0 P^{(1)} \rangle$ and $\langle l_0 P^{(1)} \rangle$ are identical; they differ only on the homothecy on $k_0 P^{(1)}$ expressed by l'_0 and l_0 respectively. Obviously, $l'_0 \approx l_0$ because $c_{01} \approx b_{01}$. Note that we omit the translation vector so that the slice lies in $sec.cell_0^{(B)}$. Thus, corresponding slices contain the same points in the lattice $L^{(0)} = Z$. This, moreover, leads to the following extension of lem. 4.

LEMMA 5. *Every maximal cell of the subdivision induced by β on $pr.cell_1^{(A)}$ corresponds to the intersection of hyperplane $\langle l'_0 P^{(1)} \rangle$, for some l'_0, with a unique maximal cell in $sec.cell_0^{(B)}$, of the same type. The cells contain the same points in $L^{(1)}$, with the same image under RC.*

PROOF. Any maximal cell in $pr.cell_1^{(A)}$ has the form $l_0 F_0 + b_{1j} + k_2 F_2 + \cdots + k_n F_n$, where faces $F_i \subset P^{(1)}, i = 0, 2, \ldots, n$, have dimensions adding up to $n-1$. Recall $pr.cell_1^{(A)}$ lies on a slice of $sec.cell_0^{(A)}$ parameterized by the value of l_0 hence, when β is employed, it gives rise to the same subdivision in every such primary cell. By construction, subspace $\langle b_{01}, F_0 \rangle$ is orthogonal and complementary to $\langle P^{(1)} \rangle$.

In $k_1 P^{(1)}$, point c_{1j} is lifted sufficiently higher than any other, so there exist maximal cells in $sec.cell_0^{(B)}$ that has it as summand. The other summands are induced by β on $\mathrm{CH}(c_{01}, k_0 P^{(1)}), k_2 P^{(1)}, \ldots, k_n P^{(1)}$. These n-dimensional cells of alg. B correspond (when intersected with $\langle l'_0 P^{(1)} \rangle$) to $(n-1)$-dimensional cells in $pr.cell_1^{(A)}$. It is straightforward to show that, for $l'_0 \in [0, k_0]$ and any β-induced cell in this Minkowski sum, its intersection with $\langle l'_0 P^{(1)} \rangle$ is a β-induced cell in $l'_0 P^{(1)} + k_2 P^{(1)} + \cdots + k_n P^{(1)}$

There exists $l'_0 \approx l_0$ that establishes the lemma, because β is applied to $(n-1)$-dimensional Minkowski sums which are almost identical, and the effect of b_{1j} and c_{1j} is the same in what concerns the lattice points in corresponding cells, following the proof of lem. 4. □

In each $sec.cell_0^{(B)}$ we distinguish 2 types of cells: cells in

$$pr.cell_1^{(B)} := \mathrm{CH}(c_{01}, k_0 P^{(1)}) + c_{1j} + k_2 P^{(1)} + \cdots + k_n P^{(1)}, \qquad (12)$$

which, by lem. 5, contains exactly the integer points in all primary cells of alg. A of the form (10) (for each slice/coset), and for each facet $P^{(2)}$ of $P^{(1)}$, cells in

$$sec.cell_1^{(B)} := \mathrm{CH}(c_{01}, k_0 P^{(2)}) + \mathrm{CH}(c_{1j}, k_1 P^{(2)}) + k_2 P^{(2)} + \cdots + k_n P^{(2)}. \qquad (13)$$

Note that both $pr.cell_1^{(B)}$ and $sec.cell_1^{(B)}$ are n-dimensional, whereas $pr.cell_1^{(A)}$ and $sec.cell_1^{(A)}$ are $(n-1)$-dimensional.

Remark 4. Every maximal cell in $sec.cell_1^{(B)}$ must have summands $F_0 = \mathrm{CH}(c_{01}, G_0), F_1 = \mathrm{CH}(c_{1j}, G_1)$, for some $G_0 \subset k_0 P^{(2)}$ and $G_1 \subset k_1 P^{(2)}$.

A similar argument as in lem. 5, implies that (13) contains exactly the integer points in the union of all secondary cells (11) defined over the various values of $l_0 \in [0, k_0]$, for a given j. The recursion steps of alg. A, for $t \geq 2$ are defined over a chain of facets $P^{(2)} \supset P^{(3)} \supset \cdots \supset P^{(n-1)}$. Hence, every $pr.cell_t^{(A)}$, for $t > 1$, contains integer points in $sec.cell_1^{(B)} \cap Z$. Therefore, we generalize the correspondence between the two algorithms by focusing on $sec.cell_1^{(B)}$.

LEMMA 6. (Main) *Every maximal cell of the subdivision induced by β on $pr.cell_t^{(A)}$, for $t \geq 2$, corresponds to the intersection of hyperplane $\langle l'_{t-1} P^{(t)} \rangle$, for some $l'_{t-1} \approx l_{t-1} \in [0, k_{t-1}] \cap \mathbb{Q}$, with a unique maximal cell in $sec.cell_1^{(B)}$, of the same type. The cells contain the same points in lattice $L^{(t)}$ with the same image under RC.*

PROOF. Primary cells of step t lie on $(n-t)$-dimensional slices of the $(n-t+1)$-dimensional $sec.cell_{t-1}^{(A)}$, parameterized by the value of $l_{t-1} \in [0, k_{t-1}]$:

$$l_0 P^{(t)} + \cdots + l_{t-1} P^{(t)} + k_t P^{(t)} + \cdots + k_n P^{(t)}. \qquad (14)$$

Similarly to rem. 3, let $l_0, \ldots, l_{t-1},\ l_i \in [0, k_i] \cap \mathbb{Q}$, define the homothecies on the first t summands of (14) and the corresponding hyperplanes $\langle l_0 P^{(t)} \rangle, \ldots, \langle l_{t-1} P^{(t)} \rangle$. Note, that $pr.cell_t^{(A)}$ is a subset of (14) and is subdivided by β into maximal cells of the form (3).

Intersecting $sec.cell_1^{(B)}$ with the above hyperplanes, yields a $(n-t)$-dimensional subset:

$$l'_0 P^{(t)} + \cdots + l'_{t-1} P^{(t)} + k_t P^{(t)} + \cdots + k_n P^{(t)}. \qquad (15)$$

This subset can also be obtained by directly intersecting $sec.cell_1^{(B)}$ with $\langle l_{t-1}P^{(t)}\rangle$. Now, $l'_i \approx l_i$, for $i = 0, 1, \ldots, t-1$ because $c_{ij} \approx b_{ij}$. For $i = 0, \ldots, t-1$, each l'_i defines a hyperplane $\langle l'_i P^{(t)}\rangle$ identical to $\langle l_i P^{(t)}\rangle$, except on the homothecy on the i-th summand. Hence, (15) is very similar to (14) in the sense that they contain the same integer points in $L^{(t)}$ and their volumes differ infinitesimally.

By def. 3 there exist n-dimensional cells in $sec.cell_1^{(B)}$ which have c_{tj} as a summand. The intersection of each of these cells with (15) shall also have c_{tj} as a summand, because this is the only point lifted highest in $P^{(t)}$. These cells correspond to the primary cell wrt alg. A of the slice (14). Moreover, this intersection is a β-induced cell in (15):

$$l'_0 F_0 + \cdots + l'_{t-1}F_{t-1} + c_{tj} + k_{t+1}F_{t+1} + \cdots + k_n F_n, \quad (16)$$

which contains the same integer points as (3). Since β is applied on $(n-t)$-dimensional polytopes which are almost identical, both (3) and (16) are of the same type. □

Corollary 7. *Using the notation of lem. 3, in particular for t-mixed cells of alg. A in the form of (4) and t, j, a t-mixed cell of alg. B is of the form:*

$$k_0E_0 + \cdots + k_{t-1}E_{t-1} + c_{tj} + k_{t+1}E_{t+1} + \cdots + k_nE_n + \delta_t \cap L,$$

where E_i is the projection of an edge of Q^β,

(a) $\langle E_0, \ldots, E_{t-1}\rangle$ is a t-dimensional space complementary to $\langle P^{(t)}\rangle$, and for $i < t$, $k_iE_i = (c_{ij}, k_ip_i)$, where $p_i \in P^{(i)}$ in lem. 3, and

(b) edges $E_{t+1}, \ldots, E_n$ are the same as in lem. 3,(4).

Proof. For $t = 0$, the corollary follows from rem. 2.

All 1-mixed cells wrt alg. B lie in (12), since every maximal cell in it has c_{1j} as a summand. By lem. 5, edges $k_2E_2, \ldots, k_nE_n$ span the $(n-1)$-dimensional space $\langle P^{(1)}\rangle$. Hence, edge k_0E_0 has to be of the form (c_{01}, k_0p_0), where $p_0 \in P^{(1)}$, by lem. 5, is as in lem. 3,(4).

Similarly, lem. 6 implies that for $t > 1$, the last $(n-t)$ edges of any t-mixed cell wrt alg. B span the $(n-t)$-dimensional space $\langle P^{(t)}\rangle$, because β induces the same subdivision on the last $n-t$ summands of (14) and (15). For the cell to be maximal, $\langle k_0E_0, \ldots, k_{t-1}E_{t-1}\rangle$ must be a t-dimensional space complementary to $\langle P^{(t)}\rangle$. By construction (see proof of lem. 6), each k_iE_i, for $i < t$, is an edge in $\mathrm{CH}(c_{ij}, k_iP^{(t)})$ of the form (c_{ij}, k_ip_i), where $p_i \in P^{(t)}$ is as in lem. 3,(4). □

Wen now consider non-mixed cells, by extending cor. 7:

Corollary 8. *Consider any non-mixed cell of alg. A, which has the form of (3) in lem. 3. It corresponds to cell:*

$$CH(c_{01}, k_0F_0) + \cdots + CH(c_{(t-1)j}, k_{t-1}F_{t-1}) + c_{tj} \\ + k_{t+1}F_{t+1} + \cdots + k_nF_n,$$

which is a non-mixed cell defined by β, where

(a) the $F_0, \ldots, F_{t-1}$ are projections of faces in Q^β, for $i < t$, and $\langle CH(c_{01}, k_0F_0), \ldots, CH(c_{(t-1)j}, k_{t-1}F_{t-1})\rangle$ is a t-dimensional space complementary to $\langle F_{t+1}, \ldots, F_n\rangle$,

(b) $F_0, \ldots, F_{t-1}, F_{t+1}, \ldots, F_n$ are the same in both cells.

We have shown that each row of the constructed matrices, indexed by points of $\mathcal{E}$ lying in a mixed or non-mixed cell, is identical for both algorithms, where $\mathcal{E}$ is the same pointset for both algorithms.

Theorem 9. *The Macaulay-type formula for the sparse resultant of generalized unmixed systems, constructed by the global lifting of sec. 2, and that constructed by D'Andrea's approach [6] are identical.*

As a consequence of thm. 9 and [6, thm. 3.8], follows thm. 2.

5. TOWARDS MIXED SYSTEMS

In studying systems with different Newton polytopes, we need the following:

Definition 5. The set of polytopes $Q_1, \ldots, Q_h \subset \mathbb{R}^n$, s.t. $\dim(\langle Q_1, \ldots, Q_h\rangle) = h-1$, is *essential* if every subset of cardinality $j, 1 \le j < h$ spans a space of dimension $\ge j$.

The sparse resultant is well defined only for essential sets of Newton polytopes. An essential set defines a Minkowski sum of dimension $h-1$ but the converse is not always true.

Alg. A admits one main modification in the mixed case: At the Recursion Phase, the faces $F_i \subset Q_i$ supported by vector v are not always the same. Let us describe the 0-th iteration for simplicity. We assume there is no additional polytope. Consider the n-dimensional secondary cell:

$$\mathrm{CH}(b_{01}, F_0) + F_1 + \cdots + F_n \subset \mathbb{R}^n,$$

where $F_i \subset \mathbb{R}^{n-1}$. Wlog, let $\{F_1, \ldots, F_k\}$ be an essential subset and let $L_+(k)$ be the integer lattice it defines. The algorithm recurses on lattice $L_+(k)$ and polytope set (representing a piece)

$$\begin{aligned} &\mathrm{CH}(b_{01}, F_0) \cap \Lambda_+(k), F_1, \ldots, F_k, \\ &F_{k+1} \cap \Lambda_+(k), \ldots, F_n \cap \Lambda_+(k), \end{aligned} \quad (17)$$

where $\Lambda_+(k)$ ranges over all possible homothetic copies of $L_+(k)$ defined by the different cosets of $L_+(k)$ in its saturation, and the different slices that can be defined as intersections with $\mathrm{CH}(b_{01}, F_0)$. Alg. A distinguishes two cases, according to whether there is one or more essential subsets of $\{F_1, \ldots, F_n\}$. In the former case, v and the corresponding secondary cell are called *admissible*. For non-admissible cells, all integer points are considered as non-mixed, i.e. treated as if they lied in non-mixed cells. For admissible cells, integer d_{F_v} is defined [6, sec.4] (cf [16]), and d_{F_v} pieces of the form (17) are (arbitrarily) selected. Lattice points labeled as mixed in these pieces by the recursive application of alg. A are labeled as mixed overall, the rest are non-mixed.

Reduced systems are such that, for any vector $v \in \mathbb{R}^n$, there is some $i \in \{1, \ldots, n\}$ so that the face supported by v in Q_i is a vertex. For us, it suffices that this holds for fewer v [5]. For such systems, as well as for arbitrary systems of 3 bivariate polynomials ($n = 2$), any sufficiently generic global lifting that lifts one vertex $b_{01} \in Q_0$ sufficiently high, thus β too, produces a Macaulay-type formula. The proof is subsumed by that for $n = 3$ below; cf also [5, 8].

Alg. B is modified so that def. 3 applies up to $i = n-1$. We sketch a proof that it produces the same matrix as alg. A, by extending the correlation between maximal cells, established in the unmixed case. Our proof could be extended to $n > 3$, but seems complicated; we expect that a more elegant approach is possible.

In non-admissible secondary cells of alg. A, for any n, we show both algorithms behave the same way, namely the corresponding lattice points lie in non-mixed cells of alg. B. We demonstrate the contrapositive by focusing on a mixed cell of alg. B and a corresponding secondary cell of alg. A, following lem. 6.

LEMMA 10. *Every t-mixed cell by alg. B, when intersected with a $(n-t)$-dimensional hyperplane as in lem 6, is contained in an admissible secondary cell of step $t-1$ of alg. A.*

PROOF. Any t-mixed cell of alg. B is $E_0 + \cdots + E_{t-1} + a_{tj} + E_{t+1} + \cdots + E_n$, where a_{tj} is either a vertex of Q_i or some c_{tj} in the interior of an $(n-t)$-dimensional face, and edges $E_{t+1}, \ldots, E_n$ span an $(n-t)$-dimensional space. This cell is intersected by a $(n-t)$-dimensional hyperplane, similarly to lem. 6. The intersection is contained in a t-primary cell of alg. A with t-summand b_{tj}; it lies in a piece of $(t-1)$-secondary cell

$$F_0 + \cdots + F_{t-2} + \mathrm{CH}(b_{(t-1)h}, F_{t-1}) + F_t + \cdots + F_n,$$

where the F_i are faces of the Q_i, $i = 1, \ldots, n$, supported by the same vector, with $\dim F_i \leq n-t$. We claim $\{F_t, \ldots, F_n\}$ contains a unique essential set, with cardinality $r+1$, spanning an r-dimensional space, which is defined as follows: F_t and $r \leq n-t$ faces, denoted wlog $F_{t+1}, \ldots, F_{t+r}$, where r is minimal so that $\dim H = r$, for $H = \langle F_t, \ldots, F_{t+r} \rangle$.

By hypothesis, $\dim\langle F_{t+1}, \ldots, F_n \rangle = n-t$, since a subspace is spanned by the E_i and has same dimension. So subsets indexed in $\{t+1, \ldots, n\}$ span a space of dimension at least equal to their cardinality. In addition, none of the $F_i, i > t+r$ is contained in H. So every subset indexed in $\{t, \ldots, n\}$ containing $\{t\} \cup J$, for $J \subset \{t+r+1, \ldots, n\}$, will be of cardinality $\leq r + |J|$ and span a space of dimension $r + |J|$. Hence there are no other essential subsets. ☐

For $n = 3$, all admissible secondary cells have d_{F_v} pieces, since there is no extra artificial polytope in the input of alg. A. We distinguish cases on the dimension $k-1$ of the space generated by the essential set $\{F_1, \ldots, F_k\}, 1 \leq k \leq 3$, on which the recursion of alg. A occurs:
(1) If $k-1$ is 0 or 1, the recursion is either trivial (occurs on a vertex), or corresponds to the Sylvester case.
(2) If $k-1 = 2$ and $\dim F_i = 1, i = 1, 2, 3$, the two algorithms behave similarly, since def. 3 defines points c_{2j} in the edges of Q_2 and the main lemma applies. Notice that $\dim Q_2 \geq 1$; otherwise the Q_i's would not form an essential set.
(3) If $k-1 = 2$, $\dim F_i \in \{1, 2\}$ for $i = 1, 2, 3$ and at least one face is 2-dimensional. If $\dim F_1 = 2$, then lem. 6 applies. Otherwise, $\dim F_1 = 1$ and $\dim F_2 \geq 1$. Irrespective of $\dim F_2$, the c_{2j} play the role of distinguished points and lem. 6 applies again.

6. COMPLEXITY

We analyze the worst-case asymptotic bit complexity of our algorithm and D'Andrea's, when they construct resultant matrices in sparse representation. We believe these bounds are not optimal and further work may tighten them.

Alg. B, implemented by the direct approach of [2], comprises of two main steps. First, the computation of the vertices of each Q_i is typically dominated. Second, we need RC for all $p \in \mathcal{E}$, in order to construct the Macaulay-type formula. Both steps can be reduced to linear programming with C constraints in V variables, and coefficient bitsize B. If we use a poly-time algorithm such as Karmarkar's, the bit complexity is $C^{5.5}V^2B^2$, where B depends on the bit-size of the input coordinates and of δ, δ_{ij}. It is related to the probability that the chosen perturbations are not sufficiently generic; see [2] for the full analysis.

Let m be the maximum number of vertices of the Q_i, r the total number of c_{ij}'s, and let $O^*(\cdot)$ indicate that we ignore polylog factors. The linear programs have complexity $O^*(r^2B^2) = O^*(m^nB^2)$ because r is bounded by the total number $O(m^{\lfloor n/2 \rfloor})$ of faces in Q. In an output sensitive manner, $r = O(|\mathcal{E}|)$, because the addition of every c_{ij} is made in order to handle at least one distinct point in $\mathcal{E}$. Hence, the complexity of constructing the Macaulay-type formula is $O^*(|\mathcal{E}|m^nB^2)$, or $O^*(|\mathcal{E}|^3B^2)$. These bounds hold also for matrices in dense representation. For generalized unmixed systems, one can use $|\mathcal{E}| = O(k^ne^nD)$ from [2, thm.3.10], where $k = \max_i\{k_i\}$, D is the total degree of the sparse resultant as a polynomial in the input coefficients, and e the basis of natural logarithms.

A better implementation finds RC for one point in a maximal cell, then enumerates all points in this cell in time proportional to their cardinality multiplied by a polynomial in m, n, B [12, thm.16]. The neighbours of these points which lie outside the cell will yield new cells, so as to explore the entire Minkowski sum; detecting new cells does not increase the overall complexity. If $S \leq |\mathcal{E}|$ is the number of maximal cells containing at least one lattice point, alg. B has complexity $O^*(Sm^nB^2 + |\mathcal{E}|)$, where typically, $S \ll |\mathcal{E}|$.

For alg. A, complexity is dominated by $O(|\mathcal{E}|n)$ linear programs, since every $p \in \mathcal{E}$ may require $O(n)$ of them for its image under RC to be determined. Each linear program has bit complexity $O(n^{7.5}m^2B^2)$. This process essentially decides in which slice of which secondary cell lies p. Although this subdivision contains much more cells than alg. B, the asymptotic analysis indicates that the latter may be slower. The optimal implementation for constructing the Macaulay-type formula should combine ideas from both algorithms.

7. REFERENCES

[1] J. Canny and I. Emiris. An efficient algorithm for the sparse mixed resultant. In *AAECC*, LNCS, pages 89–104, 1993.

[2] J.F. Canny and I.Z. Emiris. A subdivision-based algorithm for the sparse resultant. *J. ACM*, 47:417–451, 2000.

[3] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*, New York, 2nd ed., 2005.

[4] J. Canny and P. Pedersen. An algorithm for the Newton resultant. Tech. Rep. 1394, C.S. Dept, Cornell Univ, 1993

[5] C.D'Andrea, Lifting functions and Macaulay-style formulas: The resultant of sparse reduced systems. Manuscript, 2001.

[6] C. D'Andrea. Macaulay-style formulas for the sparse resultant. *Trans. of the AMS*, 354:2595–2629, 2002.

[7] C. D'Andrea and A. Dickenstein. Explicit formulas for the multivariate resultant. *J.Pure Appl.Algebra*, 164:59-86, 2001

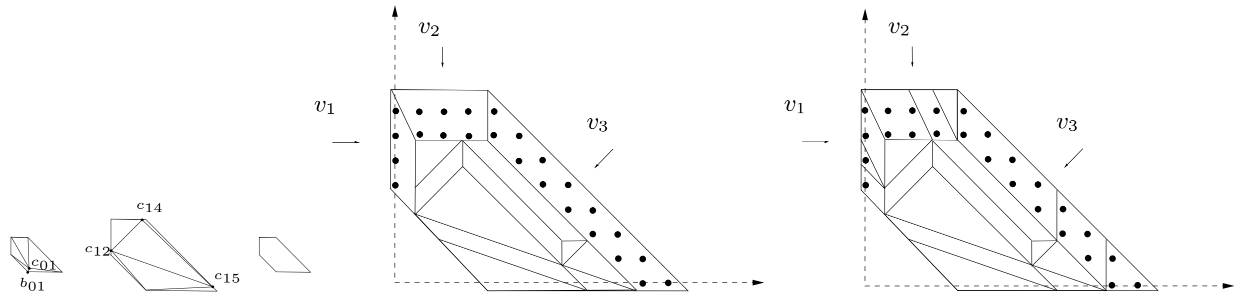
[8] C. D'Andrea, I.Z. Emiris. Lifting functions and Macaulay-style formulas for sparse resultants. Manuscript, 2003.

[9] A. Dickenstein and I.Z. Emiris. Multihomogeneous resultant formulae by means of complexes. *J. Symbolic Comput.*, 36(3-4):317–342, 2003.

[10] I.Z. Emiris and A. Mantzaflaris. Multihomogeneous resultant matrices for systems with scaled support. In *Proc. ACM ISSAC*, pages 143–150, 2009.

[11] I.Z. Emiris. *Sparse Elimination and Applications in Kinematics.* PhD thesis, UC Berkeley, 1994.

[12] I.Z. Emiris. Enumerating a subset of the integer points in a Minkowski sum, *Comp.Geom:Theory Appl*, 22:143-166,2002



(a) (b) (c)

Figure 1: Input polygons and their subdivisions wrt alg. B (fig:1a), 0-step recursion of Alg. A (fig:1b) and the mixed subdivision induced by alg. B (fig:1c).

Table 1: Illustration of cor. 7 and cor. 8

Cell wrt alg. A	Corresponding cell wrt alg.B	Type of cell
$\tilde\lambda(1,2)+(6,0)+((3,0),(1,2))+\delta_{0v3}$	$(c_{01},(1,2))+c_{15}+((3,0),(1,2))+\delta$	1-mixed
$\tilde\lambda((3,0),(1,2))+(6,0)+(3,0)+\delta_{0v3}$	$\mathrm{CH}(c_{01},(1,2),(3,0))+c_{15}+(3,0)+\delta$	non-mixed
$\tilde\lambda(1,2)+(6,0)+((3,0),(1,2))+\delta_{1v3}$	$(c_{01},(1,2))+c_{15}+((3,0),(1,2))+\delta$	1-mixed
$\tilde\lambda((3,0),(1,2))+(6,0)+(3,0)+\delta_{1v3}$	$\mathrm{CH}(c_{01},(1,2),(3,0))+c_{15}+(3,0)+\delta$	non-mixed
$\tilde\lambda(0,2)+(2,4)+((0,2),(1,2))+\delta_{0v2}$	$(c_{01},(0,2))+c_{14}+((0,2),(1,2))+\delta$	1-mixed
$\tilde\lambda((0,2),(1,2))+(2,4)+(1,2)+\delta_{0v2}$	$\mathrm{CH}(c_{01},(1,2),(0,2))+c_{14}+(1,2)+\delta$	non-mixed

[13] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston, 1994.

[14] A. Khetan. The resultant of an unmixed bivariate system. *J. Symbolic Computation*, 36:425–442, 2003.

[15] F.S. Macaulay. Some formulae in elimination. *Proc. London Math. Soc.*, 1(33):3–27, 1902.

[16] M. Minimair. Sparse resultant under vanishing coefficients. *J. Alg. Combin.*, 18:53–73, 2003.

[17] B. Sturmfels. On the Newton polytope of the resultant. *J. of Algebr. Combinatorics*, 3:207–236, 1994.

[18] H. Zhang. Calculs de résidus toriques. *C.R. Acad. Sci. Paris*, pages 639–634, 1998.

Appendix: Example

Let $n=2$, $A_0=A_2=\{(1,0),(0,1),(0,2),(1,2),(3,0)\}$, and $A_1=\{(2,0),(0,2),\ (0,4),(2,4),(6,0)\}$, so the lattice generated is $\mathbb{Z}^2$. $k_0=k_2=1$, $k_1=2$. Now $v_1=(-1,0)$, $v_2=(0,-1)$, $v_3=(-1,-1)$. Let $\delta=(-1/30,-1/30)$, (fig. 1a).

Alg. B: Let $b_{01}:=(1,0)\in Q_0$, $b_{12}:=(0,2)$, $b_{14}:=(2,4)$, $b_{15}:=(6,0)\in Q_1$, $\delta_{01}=(1/1000,1/1500)$, $\delta_{12}=(0,1/2000)$, $\delta_{14}=(-1/3000,0)$, $\delta_{15}=(-1/2000,1/2000)$. Consider integer points and their cells (fig. 1c):

point	cell in secondary cell wrt v_2	type
$(1,7),(2,7)$	$(c_{01},(0,2))+((0,4),c_{14})+(0,2)+\delta$	2
$(3,7)$	$(c_{01},(0,2))+c_{14}+((0,2),(1,2))+\delta$	1

where the summands come from Q_0,Q_1, Q_2 resp. The two cells together with cell $\sigma=\mathrm{CH}(c_{01},(0,2),(1,2))+c_{14}+(1,2)+\delta$, and some infinitesimal cells which do not contain any integer points, belong to the secondary cell wrt to v_2 of alg. A, which contains the same integer points. Points $(1,7),(2,7),(3,7)$ correspond (via an appropriate translation) to points of a piece of the secondary cell on which alg. A recurses. Cell σ does not contain any integer points because of the choice of δ_{ij}, δ.

Consider points corresponding to a piece of the secondary cell wrt to v_3, of alg. A, and their cells by β:

point	cell in secondary cell wrt v_3	type
$(4,7),(5,6),$ $(6,5),(7,4)$	$(c_{01},(1,2))+(c_{15},c_{14})+(1,2)+\delta$	2
$(8,3),(9,2)$	$(c_{01},(1,2))+c_{15}+((3,0),(1,2))+\delta$	1
$(10,1),(11,0)$	$\mathrm{CH}(c_{01},(3,0),(1,2))+c_{15}+(3,0)+\delta$	non

Consider a piece of the secondary cell wrt to v_1, of alg. A. Points in it lie in the following cells of alg. B:

point	cell in secondary cell wrt v_1	type
$(0,4)$	$(c_{01},(0,1))+c_{12}+((0,1),(0,2))+\delta$	1
$(0,5)$	$\mathrm{CH}(c_{01},(0,1),(0,2))+c_{12}+(0,2)+\delta$	non
$(0,6),(0,7)$	$(c_{01},(0,3))+(c_{12},(0,4))+(0,2)+\delta$	2

Alg. A: b_{01} is lifted to 1, all other vertices of all polygons are lifted to 0. This partitions $Q_0+Q_1+Q_2$ into a primary cell $b_{01}+Q_1+Q_2$ and 3 secondary cells corresponding to v_1,v_2,v_3, normals to the facets of Q_0 not containing b_{01}. The Q_1,Q_2 are lifted using β, which subdivides the primary cell (fig. 1b). This subdivision "coincides" with the restriction in $c_{01}+Q_1+Q_2$ of the subdivision by β, except that the latter uses c_{01} whereas the former uses b_{01}, i.e. the integer points in both subdivisions are the same and are assigned the same RC.

- We study the Recursion Phase on secondary cell:

$$\mathcal{F}_{v_1}=\mathrm{CH}(b_{01},k_0F_{v_1})+k_1F_{v_1}+k_2F_{v_1},$$

defined by facet $F_{v_1}=((0,1),(0,2))\subset Q$ supported by v_1.

$A_{1v_1}=\{(0,2),(0,4)\},A_{2v_1}=\{(0,1),(0,2)\}$, the lattice generated by $A_{1v_1}+A_{2v_1}$ is $L_+:=\langle(0,3),(0,4)\rangle\cong L_{v_1}\cong\mathbb{Z}$. The index of L_+ in L_{v_1} is $\mathrm{ind}_{v_1}=1$ and the coset representative for L_+ in L_{v_1} is $q_0=(0,0)$. The v_1-*lattice diameter* is $d_{v_1}=1$. Hence there is one slice corresponding to one piece.

We describe the recursion step on this piece. It contains points corresponding to $(0,4),(0,5)$, $(0,6)$, $(0,7)$ lying on the slice of $\mathcal{F}_{v_1}+\delta$ of the form

$$(\tilde\lambda k_0F_{v_1}+\delta')+k_1F_{v_1}+k_2F_{v_1}+\lambda F_{v_1}+\delta.$$

To define the piece, following notation in [6], the scalar multiple of F_{v_1} is $\tilde\lambda F_{v_1}=\frac{29}{30}F_{v_1}$ and the translation vector is $\delta':=(\frac{1}{30},0)$. Since we do not use an initial additional polytope, $\lambda=0$ and $\lambda_{v_1}:=\lambda+\tilde\lambda=\frac{29}{30}$.

Let $\delta_\lambda := \delta + \delta' = (0, -\frac{1}{30})$, and $\delta_\lambda = \delta_\lambda^{v_1} + \delta_{\lambda v_1}$, where $\delta_\lambda^{v_1} = (0,0) \in \mathbb{Q}v_1$ and $\delta_{\lambda v_1} = (0, -\frac{1}{30}) \in L_+ \otimes \mathbb{Q}$, hence $\delta_{0v_1} := \delta_{\lambda v_1} - q_0 = (0, -\frac{1}{30})$. So, the slice of $\mathcal{F}_{v_1} + \delta$ is

$$k_1 F_{v_1} + k_2 F_{v_1} + \lambda_{v_1} k_0 F_{v_1} + \delta_\lambda, \tag{18}$$

and the corresponding piece in L_+ is

$$k_1 F_{v_1} + k_2 F_{v_1} + \lambda_{v_1} k_0 F_{v_1} + \delta_{0v_1}. \tag{19}$$

The bijection between points in (18) and (19) is

$$p = \bar{p} + \delta_\lambda^{v_1} + q_0 = \bar{p},$$

where $p \in (18)$ and $\bar{p} \in (19)$. After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0} := k_1 F_{v_1}$, $\overline{Q_1} := k_2 F_{v_1}$, and $\overline{Q_2} := \frac{29}{30} k_0 F_{v_1}$ which is the additional polytope,
- the lattice $L_+ := \langle (0,3), (0,4) \rangle$ and
- the perturbation vector $\overline{\delta_0} := \delta_{0v_1} = (0, -\frac{1}{30})$.

In order to be compatible with β, we choose $\overline{b_{01}} = b_{12} = (0,2)$ and apply the primary lifting. This partitions $\overline{Q_0} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ into a primary $\overline{b_{01}} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ and a secondary cell $\overline{Q_0} + (0,2) + \frac{29}{30}(0,2) + \overline{\delta_0}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}} + (0,1) + \overline{Q_2} + \overline{\delta_0}$ and $\overline{b_{01}} + \overline{Q_1} + \frac{29}{30}(0,1) + \overline{\delta_0}$. The former is non-mixed and contains point $(0,5)$, corresponding to the same point on the slice, which is also non-mixed under alg. B. The latter cell is $\overline{0}$-mixed, hence 1-mixed and contains point $(0,4)$, corresponding to the same point on the slice, which is also 1-mixed under alg. B. The secondary cell $\overline{Q_0} + (0,2) + \frac{29}{30}(0,2) + \overline{\delta_0}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(0,6), (0,7)$ corresponding to the same points on the slice. They are also 2-mixed under alg. B.

- We apply recursion on secondary cell:

$$\mathcal{F}_{v_2} = \mathrm{CH}(b_{01}, k_0 F_{v_2}) + k_1 F_{v_2} + k_2 F_{v_2},$$

defined by the facet $F_{v_2} = ((0,2),(1,2))$ of Q supported by v_2.

$A_{1v_2} = \{(0,4),(2,4)\}, A_{2v_2} = \{(0,2),(1,2)\}$ and the lattice generated by $A_{1v_2} + A_{2v_2}$ is $L_+ := \langle (0,6),(1,6) \rangle \cong L_{v_2} \cong \mathbb{Z}$. The index of L_+ in L_{v_2} is $\mathrm{ind}_{v_2} = 1$ and the coset representative for L_+ in L_{v_2} is $q_0 = (0,0)$. The v_2-*lattice diameter* is $d_{v_2} := b_{01} \cdot v_2 - \min_{p \in \mathrm{CH}(b_{01} k_0 F_{v_2})} p \cdot v_2 = 2$. Hence, there are two slices, each containing one piece, and the algorithm recurses on each such piece.

We analyze the recursion step on the piece of the shifted secondary cell $\mathcal{F}_{v_2} + \delta$, which contains the integer points corresponding to the points $(1,7), (2,7), (3,7)$ lying on a slice of the shifted secondary cell $\mathcal{F}_{v_2} + \delta$ of the form

$$(\tilde{\lambda} k_0 F_{v_2} + \delta') + k_1 F_{v_2} + k_2 F_{v_2} + \lambda F_{v_2} + \delta.$$

To define this piece we have that F_{v_2} is $\tilde{\lambda} F_{v_2} = \frac{31}{60} F_{v_2}$ and the translation vector $\delta' := (\frac{29}{60}, 0)$. Now $\lambda = 0$ and hence $\lambda_{v_2} := \lambda + \tilde{\lambda} = \frac{31}{60}$. Let $\delta_\lambda := \delta + \delta' = (\frac{9}{29}, -\frac{1}{30})$. Then, δ_λ can be written as $\delta_\lambda = \delta_\lambda^{v_2} + \delta_{\lambda v_2}$, where $\delta_\lambda^{v_2} = (0,1) \in \mathbb{Q}v_2$ and $\delta_{\lambda v_2} = (\frac{9}{20}, -\frac{31}{30}) \in L_+ \otimes \mathbb{Q}$, hence $\delta_{0v_2} := \delta_{\lambda v_2} - q_0 = (\frac{9}{20}, -\frac{31}{30})$.

So, the slice of $\mathcal{F}_{v_2} + \delta$ is

$$k_1 F_{v_2} + k_2 F_{v_2} + \lambda_{v_2} k_0 F_{v_2} + \delta_\lambda, \tag{20}$$

and the corresponding piece in L_+ is

$$k_1 F_{v_2} + k_2 F_{v_2} + \lambda_{v_2} k_0 F_{v_2} + \delta_{0v_2}. \tag{21}$$

The bijection between points in (20) and points in (21) is

$$p = \bar{p} + \delta_\lambda^{v_2} + q = \bar{p} + (0,1),$$

where $p \in (20)$ and $\bar{p} \in (21)$.

After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0} := k_1 F_{v_2}$, $\overline{Q_1} := k_2 F_{v_2}$, and $\overline{Q_2} := \frac{31}{60} k_0 F_{v_2}$ which is the additional polytope,
- the lattice $L_+ := \langle (0,6), (1,6) \rangle$ and
- the perturbation vector $\bar{\delta} := \delta_{0v_2} = (\frac{9}{20}, -\frac{31}{30})$.

To be compatible with β, we choose $\overline{b_{01}} = b_{14} = (2,4)$ and apply the primary lifting; this partitions the Minkowski sum $\overline{Q_0} + \overline{Q_1} + \overline{Q_2} + \bar{\delta}$ into a primary $\overline{b_{01}} + \overline{Q_1} + \overline{Q_2} + \bar{\delta}$ and a secondary cell $\overline{Q_0} + (0,2) + \frac{31}{60}(0,2) + \bar{\delta}$. Lifting β induces a mixed subdivision of the primary cell consisting of the cells $\overline{b_{01}} + (1,2) + \overline{Q_2} + \bar{\delta}$ and $\overline{b_{01}} + \overline{Q_1} + \frac{31}{60}(0,2) + \bar{\delta}$. The latter is $\overline{0}$-mixed, hence 1-mixed and contains the integer point $(3,6)$ corresponding to point $(3,7)$ on the slice which is also 1-mixed under alg. B. The former is non-mixed and does not contain any integer points.

The secondary cell $\overline{Q_0} + (0,2) + \frac{31}{60}(0,2) + \bar{\delta}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(1,6), (2,6)$ corresponding to the points $(1,7), (2,7)$ of the slice respectively; they are also 2-mixed under alg. B.

- The last secondary cell is

$$\mathcal{F}_{v_3} = \mathrm{CH}(b_{01}, F_{v_3}) + k_1 F_{v_3} + k_2 F_{v_3},$$

defined by the facet $F_{v_3} = ((3,0),(1,2))$ of Q supported by $v_3 = (-1,-1)$.

$A_{1v_3} = \{(6,0),(2,4)\}, A_{2v_3} = \{(3,0),(1,2)\}$, the lattice generated by $A_{1v_3} + A_{2v_3}$ is $L_+ := \langle (9,0),(7,2) \rangle \cong 2\mathbb{Z}$ and $L_{v_3} \cong \mathbb{Z}$. The index of L_+ in L_{v_3} is $\mathrm{ind}_{v_3} = 2$ and the cosets representatives for L_+ in L_{v_3} are $q_0 = (0,0)$ and $q_1 = (-1,1)$. The v_3-*lattice diameter* is $d_{v_3} := b_{01} \cdot v_3 - \min_{p \in \mathrm{CH}(b_{01} k_0 F_{v_3})} p \cdot v_3 = 2$. Hence there are two slices, each corresponding to two pieces, and the algorithm recurses on each such piece.

We analyze the recursion step on the two pieces that contain integer points corresponding to points $(11,0), (10,1), (9,2), (8,3), (7,4), (6,5), (5,6), (4,7)$ lying on a slice of the shifted secondary cell $\mathcal{F}_{v_3} + \delta$ of the form

$$(\tilde{\lambda} k_0 F_{v_3} + \delta') + k_1 F_{v_3} + k_2 F_{v_3} + \lambda F_{v_3} + \delta.$$

To define these pieces, we have that the scalar multiple of F_{v_3} is $\tilde{\lambda} F_{v_3} = \frac{32}{60} F_{v_3}$ and the translation vector is $\delta' := (\frac{7}{15}, 0)$. Now, $\lambda = 0$ and hence $\lambda_{v_3} := \lambda + \tilde{\lambda} = \frac{32}{60}$; Let $\delta_\lambda := \delta + \delta' = (\frac{13}{30}, -\frac{1}{30})$.

Then, δ_λ can be written as $\delta_\lambda = \delta_\lambda^{v_3} + \delta_{\lambda v_3}$, where $\delta_\lambda^{v_3} = (1,1) \in \mathbb{Q}v_3$ and $\delta_{\lambda v_3} = (-\frac{17}{30}, -\frac{31}{30}) \in L_+ \otimes \mathbb{Q}$, hence $\delta_{0v_3} := \delta_{\lambda v_3} - q_0 = (-\frac{17}{30}, -\frac{31}{30})$ and $\delta_{1v_3} := \delta_{\lambda v_3} - q_1 = (\frac{13}{30}, -\frac{61}{30})$.

So, the slice of $\mathcal{F}_{v_3} + \delta$ is

$$k_1 F_{v_3} + k_2 F_{v_3} + \lambda_{v_3} k_0 F_{v_3} + \delta_\lambda, \tag{22}$$

and the corresponding pieces in L_+ are

$$k_1 F_{v_3} + k_2 F_{v_3} + \lambda_{v_3} k_0 F_{v_3} + \delta_{0v_3}, \tag{23}$$

$$k_1 F_{v_3} + k_2 F_{v_3} + \lambda_{v_3} k_0 F_{v_3} + \delta_{1v_3}, \tag{24}$$

The correspondences between points in the slice and points in the pieces are

$$p = \bar{p} + \delta_\lambda^{v_3} + q_0 = \bar{p} + (1,1),$$

where $p \in (22)$ and $\bar{p} \in (23)$, and

$$p = \bar{p} + \delta_\lambda^{v_3} + q_1 = \bar{p} + (0,2),$$

where $p \in (22)$ and $\bar{p} \in (24)$.

After re-indexing, the input of the recursion step is:
- the polygons $\overline{Q_0} := k_1 F_{v_3}$, $\overline{Q_1} := k_2 F_{v_3}$, and $\overline{Q_2} := \frac{32}{60} k_0 F_{v_3}$ which is the additional polytope,
- the lattice $L_+ := \langle (9,0), (7,2) \rangle$ and
- the perturbation vectors $\overline{\delta_0} := \delta_{0v_3} = (-\frac{17}{30}, -\frac{31}{30})$ and $\overline{\delta_1} := \delta_{1v_3} = (\frac{13}{60}, -\frac{61}{30})$.

As β indicates, we choose $\overline{b_{01}} = b_{15} = (6,0)$ and apply the primary lifting.

For the first piece, the lifting partitions the Minkowski sum $\overline{Q_0} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ into a primary $\overline{b_{01}} + \overline{Q_1} + \overline{Q_2} + \overline{\delta_0}$ and a secondary cell $\overline{Q_0} + (1,2) + \frac{32}{60}(1,2) + \overline{\delta_0}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}} +$

$(3,0)+\overline{Q_2}+\overline{\delta_0}$ and $\overline{b_{01}}+\overline{Q_1}+\frac{32}{60}(1,2)+\overline{\delta_0}$. The former is non-mixed and contains point $(9,0)$, which corresponds to $(10,1)$ on the slice which is also non-mixed under alg. B. The latter is $\overline{0}$-mixed, hence 1-mixed and contains the point $(7,2)$ corresponding to the point $(8,3)$ in the slice which is also 1-mixed under alg. B.

The secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_0}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(3,6),(5,4)$ corresponding to the points $(4,7),(6,5)$ of the slice respectively which are also 2-mixed under alg. B.

For the second piece, the lifting partitions the Minkowski sum $\overline{Q_0}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_1}$ into a primary $\overline{b_{01}}+\overline{Q_1}+\overline{Q_2}+\overline{\delta_1}$ and a secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_1}$. Lifting β induces a mixed subdivision on the primary cell consisting of the cells $\overline{b_{01}}+(3,0)+\overline{Q_2}+\overline{\delta_1}$ and $\overline{b_{01}}+\overline{Q_1}+\frac{32}{60}(1,2)+\overline{\delta_1}$. The former is non-mixed and contains point $(11,-2)$ corresponding to $(11,0)$ on the slice which is also non-mixed under alg. B, whereas the latter cell is $\overline{0}$-mixed, hence 1-mixed and contains the integer point $(9,0)$ corresponding to point $(9,2)$ on the slice which is also 1-mixed under alg. B.

The secondary cell $\overline{Q_0}+(1,2)+\frac{32}{60}(1,2)+\overline{\delta_1}$ is $\overline{1}$-mixed, hence 2-mixed and contains the integer points $(7,2),(5,4)$ corresponding to the points $(7,4),(5,6)$ of the slice respectively. These are also 2-mixed under alg. B. Table 6 illustrates cor. 7 and 8, where the summands come from Q_0, Q_1 and Q_2 respectively. Recall that $c_{01} := (1,0)+\delta_{10}$, $c_{14} := (2,4)+\delta_{14}$ and $c_{15} := (6,0)+\delta_{15}$.