Graphs with flexible labelings

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(joint work with Josef Schicho and Georg Grasegger)

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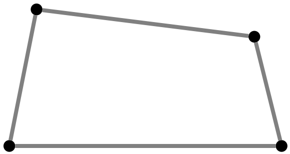
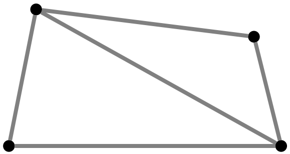
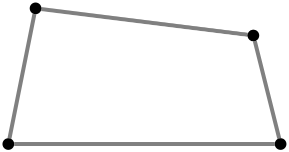
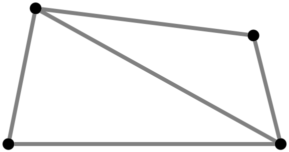
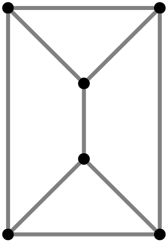
EρΓA seminar, September 21, 2017

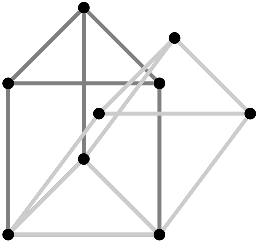


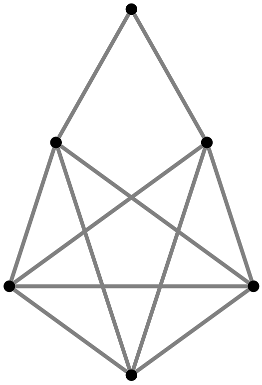


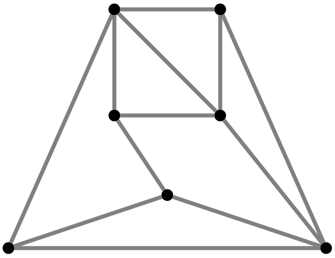












Flexible labelings of graphs

Definition

Let $G = (V, E)$ be a simple connected graph with at least one edge.

- Let $\lambda : E \to \mathbb{R}_+$ be an edge labeling of G.
  A map $\rho : V \to \mathbb{R}^2$ is a *realization* of G compatible with λ iff

  $$\|\rho(u) - \rho(v)\| = \lambda(uv)$$

  for all edges uv in E.
- Two realizations ρ_1 and ρ_2 are equivalent iff there exists a direct Euclidean isometry σ of $\mathbb{R}^2$ such that $\rho_1 = \sigma \circ \rho_2$.

Definition

A labeling λ of G is:

- *realizable* if there is a realization of G compatible with λ,
- *rigid* if it is realizable and the number of realizations of G compatible with λ up to equivalence is finite,
- *flexible* if the number of realizations of G compatible with λ up to equivalence is infinite.

Equivalently: Let $\bar{u}\bar{v}$ be an edge of G and $C \subset (\mathbb{R} \times \mathbb{R})^V$ be the zero set of

$$\begin{aligned} (x_{\bar{u}}, y_{\bar{u}}) &= (0,0) \\ (x_{\bar{v}}, y_{\bar{v}}) &= (\lambda(\bar{u}\bar{v}), 0) \\ (x_u - x_v)^2 + (y_u - y_v)^2 &= \lambda(uv)^2, \quad \forall\, uv \in E \end{aligned} \tag{$*$}$$

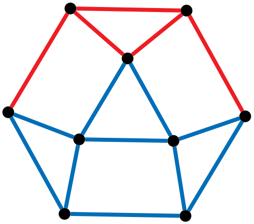
A labeling λ of G is:

- *realizable* if $|C| > 0$,
- *rigid* if $0 < |C| < \infty$,
- *flexible* if $|C| = \infty$.

NAC-colorings

Definition

Let $\delta : E \to \{\text{blue}, \text{red}\}$ be a coloring of edges.
A cycle in G is an *almost red cycle*, if exactly one of its edges is blue.
A coloring δ is called a *NAC-coloring*, if it is surjective and there are no almost blue or almost red cycles in G.



Main theorem

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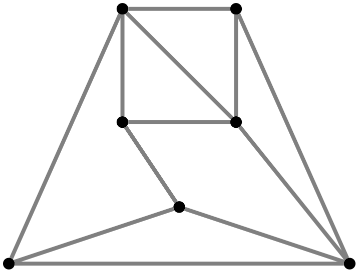

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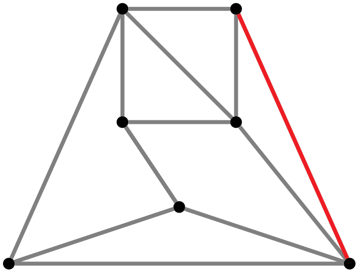

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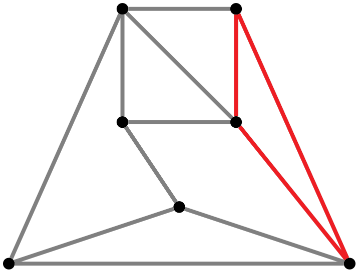

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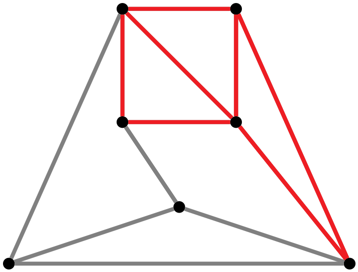

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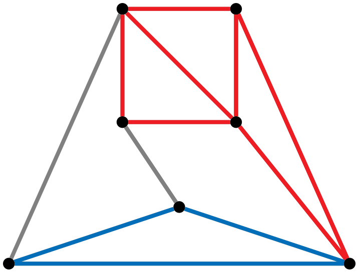

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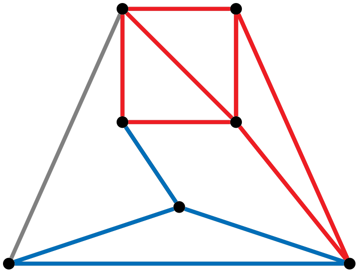

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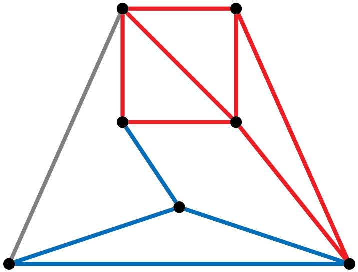

$\Longrightarrow$ no flexible labeling

Definition

A *valuation* of a field F is a nontrivial surjective mapping $\nu : F^* \to W$ such that for all $a, b \in F^*$:

i) $\nu(ab) = \nu(a) + \nu(b)$,

ii) if $a + b \in F^*$, then $\nu(a + b) \geq \min(\nu(a), \nu(b))$.

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Theorem (Chevalley)

Let F be an algebraic function field over k. If $X \in F$ is transcendental over k, then there exists at least one valuation ν of F such that $\nu(X) > 0$ and ν is trivial on k.

Proof of the main theorem

$\Longrightarrow$: Let λ be a flexible labeling of $G = (V, E)$ and C be an irreducible curve in the zero set of $(*)$ over $\mathbb{C}$.
Let F be the complex function field of C.

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Let F be the complex function field of C.
For every $uv \in E$, define $W_{uv}, Z_{uv} \in F$ by

$$W_{uv} := x_v - x_u + i(y_v - y_u)\,,$$
$$Z_{uv} := x_v - x_u - i(y_v - y_u)\,.$$

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Hence,

$$W_{uv} Z_{uv} = (x_u - x_v)^2 + (y_u - y_v)^2 = \lambda(uv)^2 \in \mathbb{R}^*\,,$$
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Take a valuation ν of F such that $\nu(W_{u'v'}) > 0$ for some $u'v' \in E$ s.t. $W_{u'v'}$ is transcendental.

Define a NAC-coloring:

$$\delta(e) = \begin{cases} \text{red if } \nu(W_e) > 0\,, \\ \text{blue if } \nu(W_e) \leq 0\,. \end{cases}$$

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For a path $(u_1, u_2, \ldots, u_n)$:

$$\begin{aligned} W_{u_1 u_n} &= x_{u_n} - x_{u_1} + i(y_{u_n} - y_{u_1}) \\ &= x_{u_2} - x_{u_1} + \cdots + x_{u_n} - x_{u_{n-1}} + i(y_{u_2} - y_{u_1} + \ldots y_{u_n} - y_{u_{n-1}}) \\ &= W_{u_1 u_2} + W_{u_2 u_3} + \ldots + W_{u_{n-1} u_n}\,. \end{aligned}$$

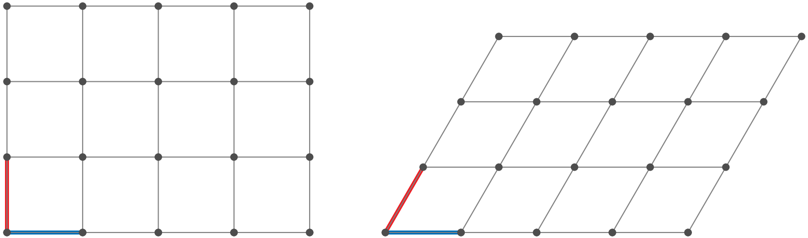
All red:

$$\nu(W_{u_1 u_n}) = \nu(W_{u_1 u_2} + W_{u_2 u_3} + \ldots + W_{u_{n-1} u_n}) \geq \min \nu(W_{u_i u_{i+1}}) > 0\,.$$

All blue:

$$\begin{aligned} \nu(W_{u_1 u_n}) &= -\nu(Z_{u_1 u_n}) = -\nu(Z_{u_1 u_2} + \ldots + Z_{u_{n-1} u_n}) \\ &\leq -\min \nu(Z_{u_i u_{i+1}}) = \max \nu(W_{u_i u_{i+1}}) \leq 0\,. \end{aligned}$$

$\Longleftarrow$: Let δ be a NAC-coloring.



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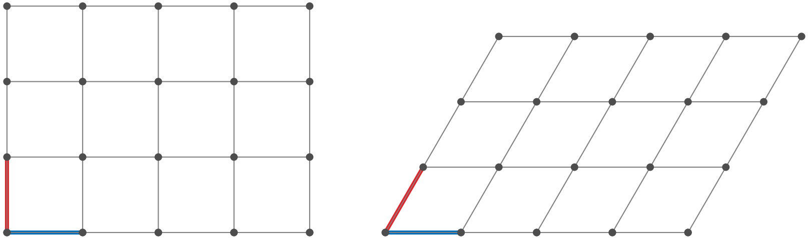
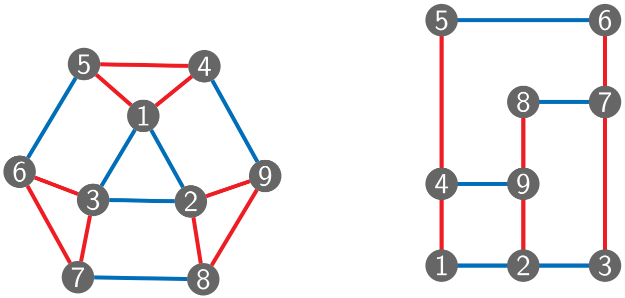


Let $R_1, \dots, R_m$, resp., $B_1, \dots, B_n$ be the connected components of G^{δ}_{red} , resp., G^{δ}_{blue}.

- Define a realization $\rho : V \to \mathbb{R}^2$ by $\rho(v) = (i, j)$, where i and j are such that $v \in R_i \cap B_j$.
- A flexible labeling λ is given by $\lambda(uv) = \|\rho(u) - \rho(v)\|$ for $uv \in E$.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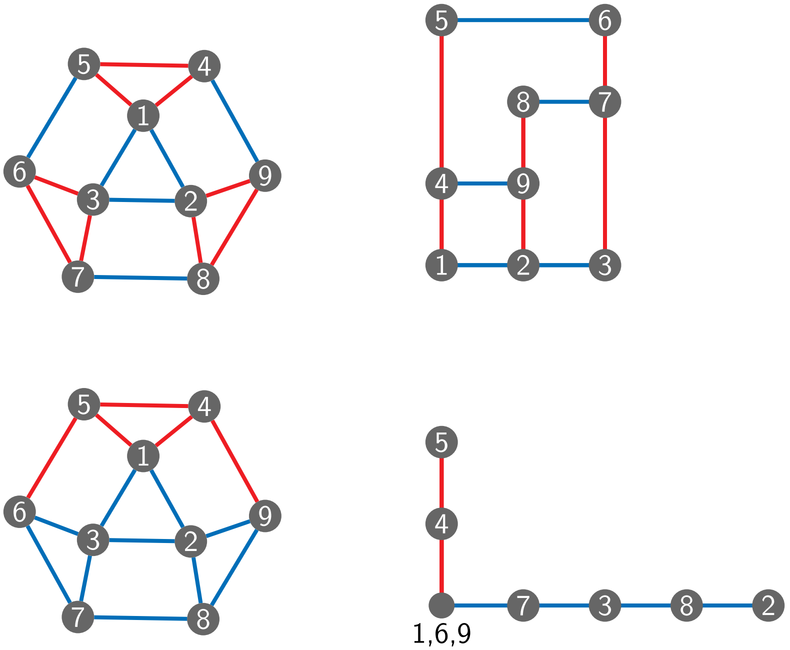
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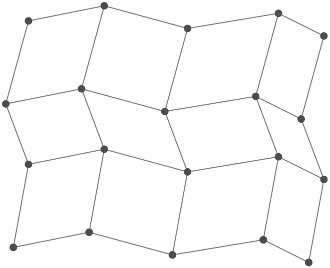

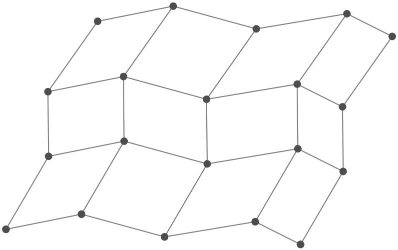

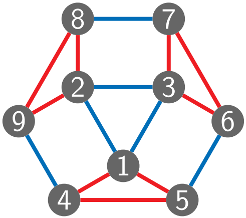
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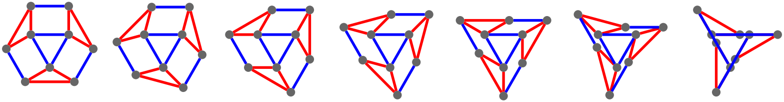

Graphs with NAC-coloring

Definition

Let $\sim'_\triangle$ be a relation on $E \times E$ such that $e_1 \sim'_\triangle e_2$ iff there exists a triangle subgraph C_3 of G such that $e_1, e_2 \in E_{C_3}$. Let $\sim_\triangle$ be the reflexive-transitive closure of $\sim'_\triangle$.

- G is called $\triangle$-*connected* if $e_1 \sim_\triangle e_2$ for all $e_1, e_2 \in E$.
- An edge e is called a *connecting edge* if it is not in $\sim_\triangle$ relation with any other edge.

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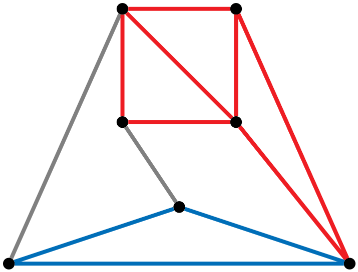

Theorem (Necessary condition)

If a graph G has a NAC-coloring, then there is no spanning subgraph H of G such that H is $\triangle$-connected.

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If there exists an independent set of vertices V_c which separates G, then G has a NAC-coloring.

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Lemma (Sufficient condition)

Let G has at least two edges. If there is a set E_c of connecting edges of G such that E_c separates G and the subgraph of G induced by E_c contains no path of length four, then G has a NAC-coloring.

Theorem (Necessary condition)

If a Laman graph G has a NAC-coloring, then G is not $\triangle$-connected.

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Laman graphs

Definition

A graph $G = (V, E)$ is called *Laman*, if $|E| = 2|V| - 3$ and $|E'| \leq 2|V'| - 3$ for every subgraph (V', E') of G.

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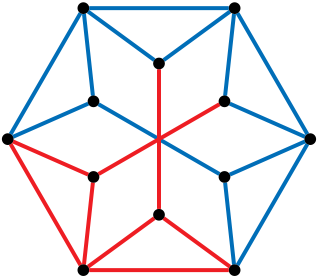
Definition

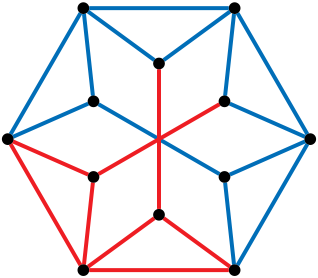
A Laman graph G is called *problematic*, if the following hold:

1. $\deg(v) \geq 3$ for all $v \in V_G$,
2. if $\deg(v) = 3$, then exactly two neighbours of v are connected by an edge and both have degree at least 4,
3. all vertices are in some triangle $C_3 \subset G$.



Theorem

Let G be a Laman graph. If G is not $\triangle$-connected, then it has a NAC-coloring, or there exists a problematic graph G' with no NAC-coloring such that G can be constructed from G' by Henneberg steps.



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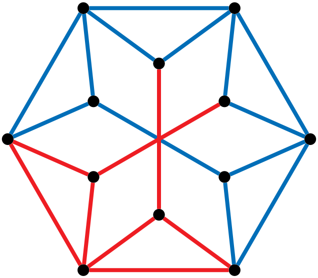
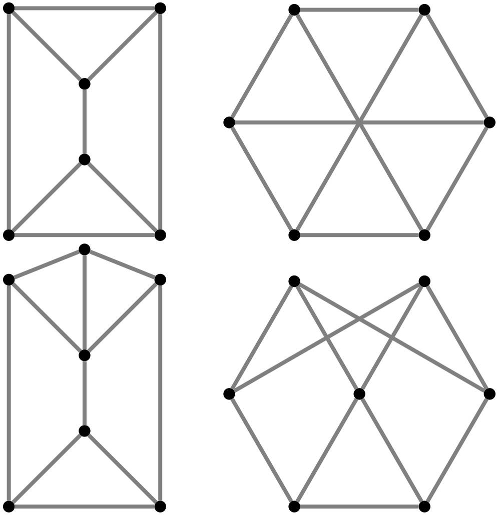
Conjecture

There is no problematic Laman graph without NAC-coloring.

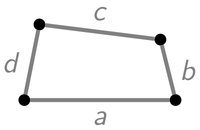


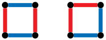
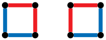
Current and future work

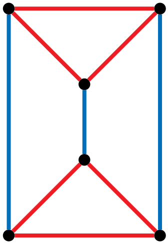
How about flexible labelings with injective realizations?

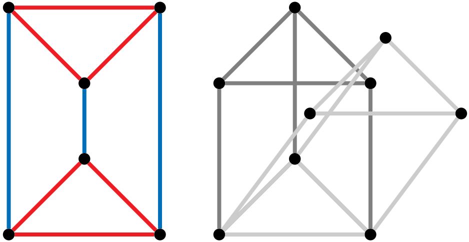


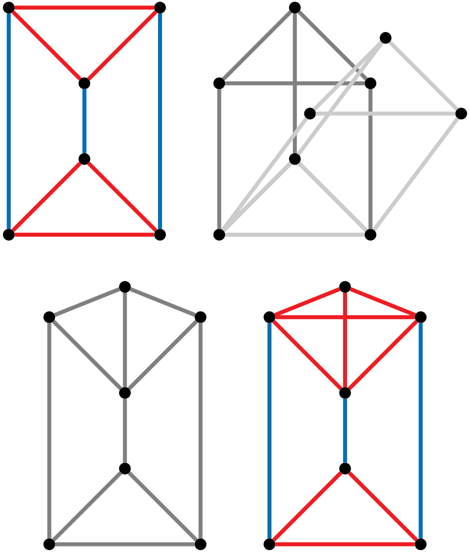
NAC-colorings of quadrilateral



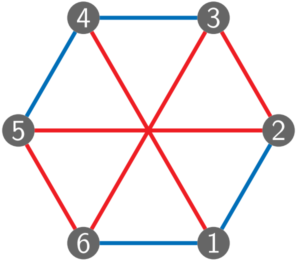
Quadrilateral	Motion	NAC-colorings	Equal lengths
General			$\emptyset$
Deltoid	nondeg.		$a = d,\ b = c$
	deg.		
Parallelogram	parallel		$a = c,\ b = d$
	antipar. or deg. rhombus		





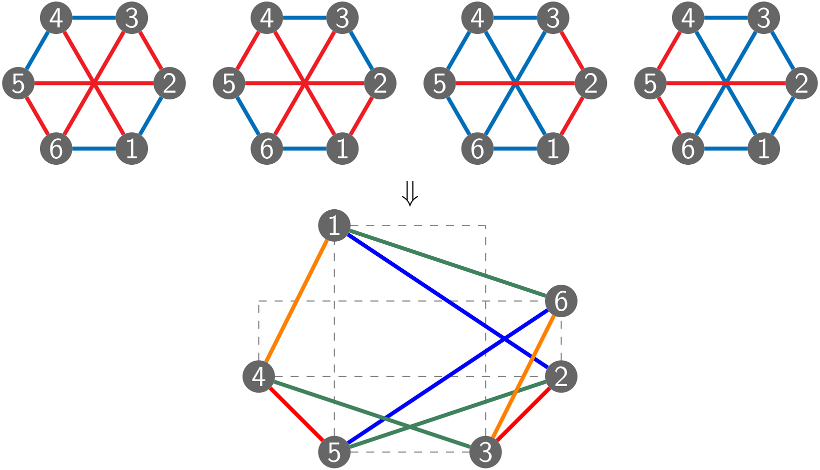


Comparing leading coefficients



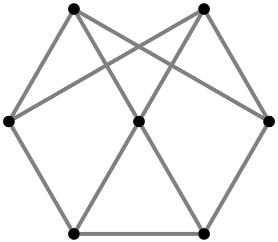
$\Longrightarrow$ Quadrilateral 2365 has perpendicular diagonals.

Ramification formula



Dixon II

Triangle inequality



Injective realizations are possible only if the triangle is degenerated.

Thank you